(Result 2) For any sentences $P_1, \ldots, P_n$, Q in M,

Q is a logical consequence of $P_1, \ldots, P_n$ iff Q is a S5-semantic consequence of $P_1, \ldots, P_n$.

The argument given for (Result 2) will rely on (Necessary Bivalence) and (Relative Possibility is S5)

(Necessary Bivalence) Necessarily, each sentence meaning is either true or false (but not both)

(Relative Possibility is S5) The relative possibility relation is trivial ~~reflexive, symmetric and transitive~~ (that is, it relates all worlds to each other)

For simplicity, the argument will also rely on the possible worlds theory of sentence meanings according to which ~~each~~ sentence meanings are functions from possible worlds to truth values, where there are only two truth values — true and false.

In order to establish (Result 2) ~~two~~ we will first establish ~~three~~ two preliminary lemmas.

Lemma 1 For any meaning interpretation m,
for any possible world u,
there is a S5-model $\langle W, R, v\rangle$
and a $w \in W$
such that, for any sentence in M:

A is true at w under $\langle W, R, v\rangle$

iff A is true at u under m

Proof of Lemma 1. Let m be a meaning interpretation of M, and let u be a possible world. Let W be the set of all possible worlds, ~~R be the set trivial set $\{\langle w, w'\rangle \mid w, w' \in W\}$~~, and let v be the function on the set of sentential constants such that, for any $w \in W$, $V_w(p) = 1$ if $m(p)$ is true at w, and $V_w(p) = 0$ if $m(p)$ is false at w. (Given ~~this~~ Necessary bivalence, this defines v.)

We now need to establish that (*) is true for every sentence A in M.

~~(*) A is true at a under m iff~~

(*) A is true at a under $\langle W, v\rangle$

iff A is true at a under m

The proof is ~~simila~~ by induction, and is similar to the proof given in the proof of (Lemma) in the argument for (Result 1). I will only discuss the inductive step for the case of necessity.

<u>The case of necessity</u>

Suppose A satisfies (*). We need to show that $\Box A$ also satisfies (*).

Since A satisfies (*), we have

(i) A is true at a under $\langle W, v\rangle$ iff

A is true at a under m

By the definition of the extension of v to all sentences in M

ii) '$\mathcal{D}A$' is true at a under $\langle W, v \rangle$ iff

for any $w \in W$, ~~such that $\langle a, w \rangle$~~

A is true at w under $\langle W, v \rangle$

~~Si~~ Given (Relative possibility is S5), it follows from the possible worlds analysis of necessity that

iii) '$\mathcal{D}A$' is true at a under m iff,

for any possible world w,

A is true at w under m

It now follows from (i–iii) and the definitions of W that '$\mathcal{D}A$' is true at a under $\langle W, v \rangle$ iff '$\mathcal{D}A$' is true at a under m. Hence '$\mathcal{D}A$' satisfies (*).

~~Given~~ (*) has now been established. Since (Lemma 1) follows from (*), (Lemma 1) is also established.

∎

(Lemma 2) For any S5 model $\langle W, V \rangle$, for any $w_0 \in W$, there is a meaning interpretation m and a possible world u_0 such that,

for any sentence A in M

A is true at u_0 under m iff

A is true at w_0 under $\langle W, V \rangle$

Proof of Lemma 2. Let $\langle W, v \rangle$ be a S5-model, and let $w_0 \in W$.

Def: For any $w, w' \in W$, define

$w \sim w'$ iff, for any sentential constant p in M, $v_w(p) = v_{w'}(p)$

Def: X is a W-class iff

i) for some $w \in W$, $w \in X$;

ii) for any $w, w' \in W$, if $w \in X$ ~~then~~ and $w \sim w'$, then $w' \in X$; and

iii) $X \subseteq W$.

Some background tacit assumptions:

(Assumptions) i) There are ~~a~~ countably infinitely many sentential constants in M

ii) There are at least countably infinitely many possible worlds

iii) There is a set of possible worlds

It follows from (Assumptions) that

a) there is at most a countable infinity of W-classes (since there are only a countable infinity of sentence constants)

b) There is a function f from the set of ~~world~~ possible worlds onto the set of W-classes

For each sentential constant p, let m(p) be the function g from the set of possible worlds to ~~truth values~~ the set {Truth, Falsity} such that, for any pos world u,

$g(u)$ = truth iff $v_w(p) = 1$ for all $w \in f(u)$.

Let u_0 be a pos world ~~at~~ such that $w_0 \in f(u_0)$.

We now need to show that, for any sentence A in M,

(**) A is true at u_0 under m iff A is true at w_0 under $\langle W, v \rangle$

on the number of operator expressions in A

The argument is by induction. I will only do the base step. The inductive step is similar to that in the proof of Lemma 1.

Base step

Suppose p is a sentential constant in M

p is true at w_0 under m ~~iff~~

iff $m(p)$ is true at w_0.

iff $g(w_0)$ = truth

iff $v_{w_0}(p) = 1$

iff p is true at w_0 under $\langle W, v \rangle$

Note: It is part of the ~~…~~ possible worlds theory of sentence meanings that

i) sentence meanings are functions from the set of possible worlds to {Truth, Falsity}, and

ii) such a function f is true at a pos world w iff $f(w)$ = truth.

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Proof of (Result 2).

I will first establish the left to right direction by contraposition. Suppose Q is not a S5-semantic consequence of $P_1, \ldots, P_n$. Then there is a S5-model $\langle W, v\rangle$ and a $w_0 \in W$ such that $P_1, \ldots, P_n$ are all true at w_0 under $\langle W, v\rangle$, but Q is false at w_0 under $\langle W, v\rangle$. By lemma 3, there is a meaning interpretation m and a possible world u_0 such that $P_1, \ldots, P_n$ are all true at u_0 under m, but Q is false at u_0 under m. Hence Q is not a logical consequence of $P_1, \ldots, P_n$. This establishes the left to right direction of (Result 2).

Suppose now that Q is not a logical consequence of $P_1 \ldots P_n$. Then there is a meaning interpretation m and a possible world u_0 such that $P_1, \ldots, P_n$ are all true at u_0 under m, but Q is false at u_0

under m. By lemma 1 there is a S5-model $\langle W, v \rangle$ and a $w_0 \in W$ such that $P_1, \ldots, P_n$ are all true ~~models~~ at w_0 under $\langle W, v \rangle$. It follows that Q is not a S5 semantic consequence of $P_1 \ldots P_n$. This establishes the right to left direction of (Result 2). Hence (Result 2) is established.

$\Box$